Bayesian hierarchical models for growth rates, debt to GDP ratios and financial stress

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November 26, 2017

Contents

1 Introduction

The goal of this paper is to examine the relationship between *growth rate of GDP* and *level of the debt to GDP ratio*. The other predictors available are *financial stress*, *growth rate of the labor force*. I also use the *growth rate lagged by one period* as a predictor. The debt to GDP ratio is also a lagged variable as the theoretical underpinning suggests that the debt to GDP ratio in the previous country quarter[1] would impact the growth rate of GDP in the current country quarter.

[1] A country quarter is a cross sectional and time period marker, so the first quarter in 1981 is one country quarter and the second quarter in 1981 is another and so on.

2 Exploratory Data Analysis

I perform some exploratory data analysis trying to capture the relationship of growth rate of GDP with each of the predictors individually to understand the dataset better and graphically represent possible trends.

2.1 Growth Rate of GDP vs. Debt to GDP Ratio

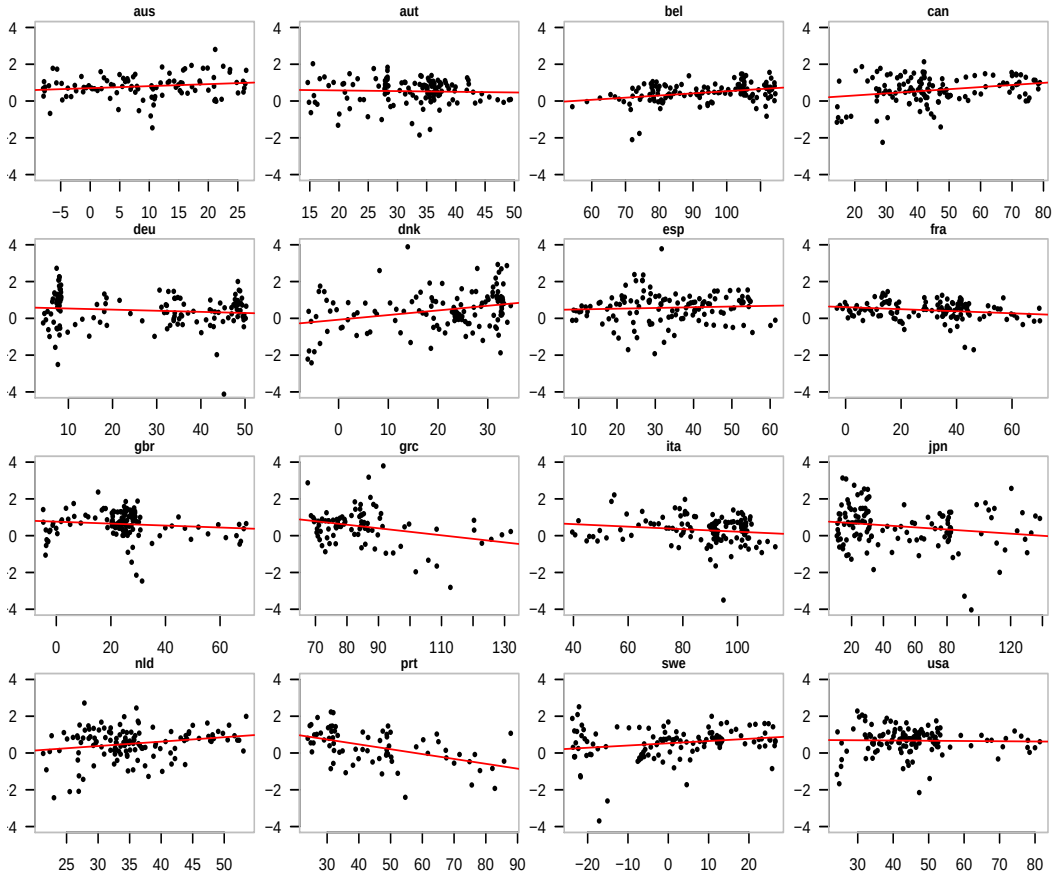

Figure 1: *The horizontal axis is the Debt to GDP Ratio. The vertical axis is the Growth Rate of GDP for each country under study. The red line is the OLS regression line.*

The horizontal axis is not on a uniform scale and this is owing to the very wide variation in the Debt to GDP Ratio. The range is approximately [-20, 140] and this makes it very difficult to plot on a uniform axis. However the goal of the plot is to merely view the relationship between the two variables and theory suggests that there may be a negative relationship, which is evident in some countries but this is not easily inferred, and further Bayesian hierarchical analysis is required to comment satisfactorily.

2.2 Growth Rate of GDP vs. Financial Stress

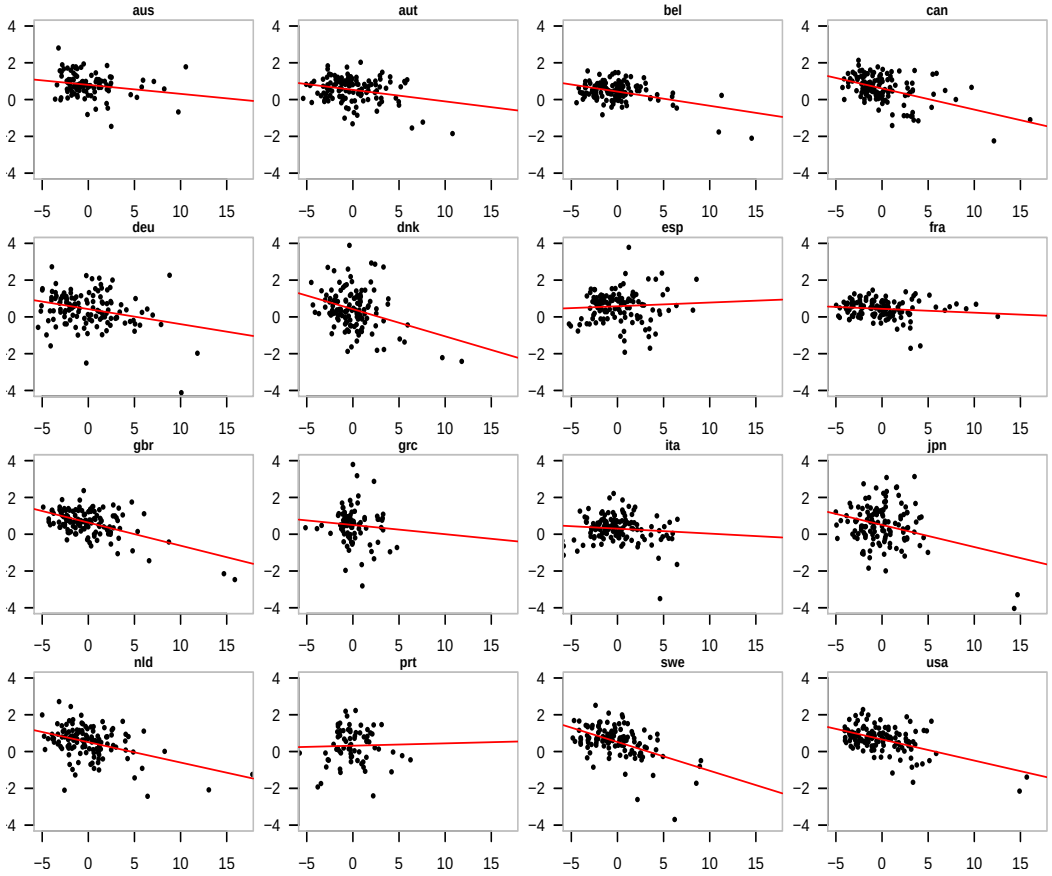

Figure 2: *The horizontal axis is the Financial Stress index. The vertical axis is the Growth Rate of GDP for each country under study. The red line is the OLS regression line.*

The Growth Rate of GDP appears to have a negative relationship with financial stress which is understandable and expected. At higher levels of financial stress we see that countries' growth rates slow down.

2.3 Growth Rate of GDP vs. Labor force growth

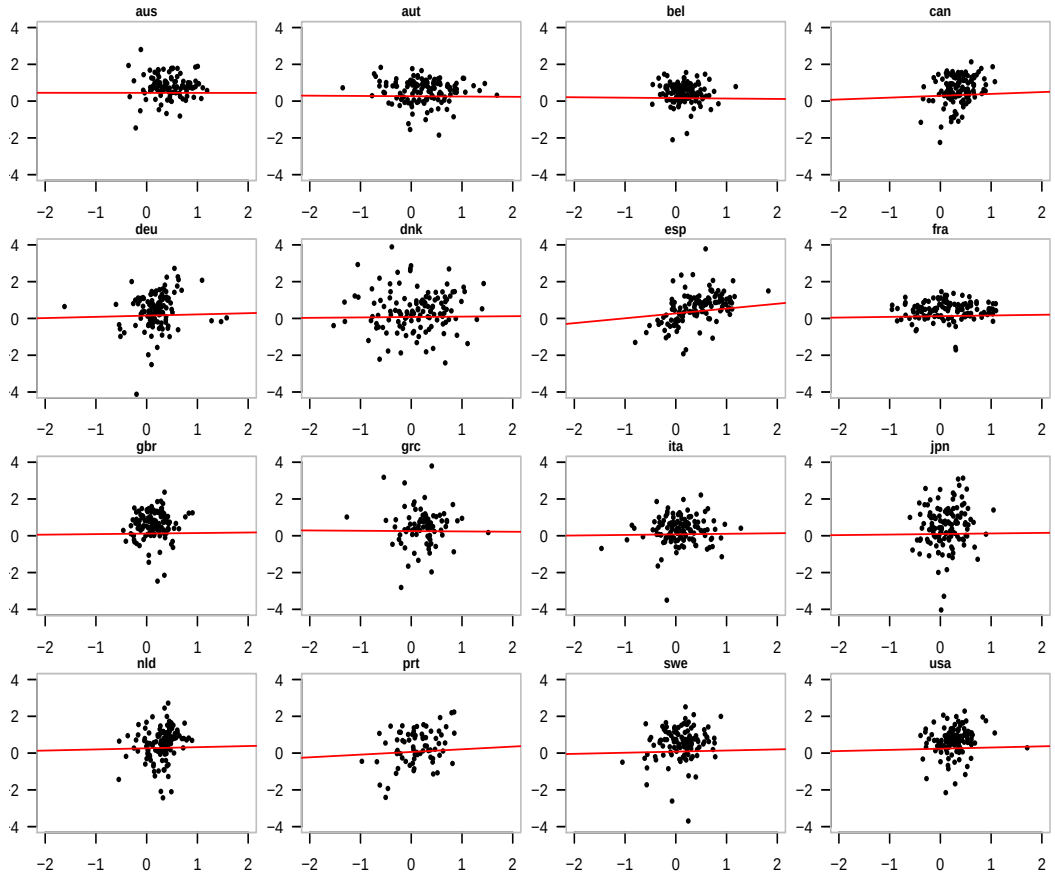

Figure 3: *The horizontal axis is the Labor force growth. The vertical axis is the Growth Rate of GDP for each country under study. The red line is the OLS regression line.*

Growth Rate of GDP would be expected to have a positive relationship with the growth of the Labor force. We see that the pattern is observed in some countries but to a very small extent.

3 Models

3.1 Model 1

$$p(y_{it}|\beta_{0i}, \beta_{yi}, \beta_{di}, \beta_{fs}, \beta_l) \sim \mathcal{N}(\beta_{0i} + \beta_{yi} \cdot y_{it-1} + \beta_{di} \cdot d_{it-1} + \beta_{fs} \cdot fs_{it} + \beta_l \cdot lab_{it}, \sigma^2) \quad (1)$$

$$p(\beta_0) \sim \mathcal{N}(\mu_0, \tau_0) \quad (2)$$

$$p(\beta_y) \sim \mathcal{N}(\mu_y, \tau_y) \quad (3)$$

$$p(\beta_d) \sim \mathcal{N}(\mu_d, \tau_d) \quad (4)$$

$$p(\beta_{fs}) \sim \mathcal{N}(\mu_{fs}, \tau_{fs}) \quad (5)$$

$$p(\beta_l) \sim \mathcal{N}(\mu_l, \tau_l) \quad (6)$$

Equation 1 is the likelihood function and equations 2 through 6 are priors on the parameters. This is a simple linear model where growth Rate of GDP in a country quarter (y_{it}) is a function of growth rate of GDP in the previous period (y_{it-1}), debt ratio in the previous period (d_{it-1}), financial stress index value (fs_{it}) and labor force growth rate (lab_{it}).

Posterior Predictive Checks - Model 1

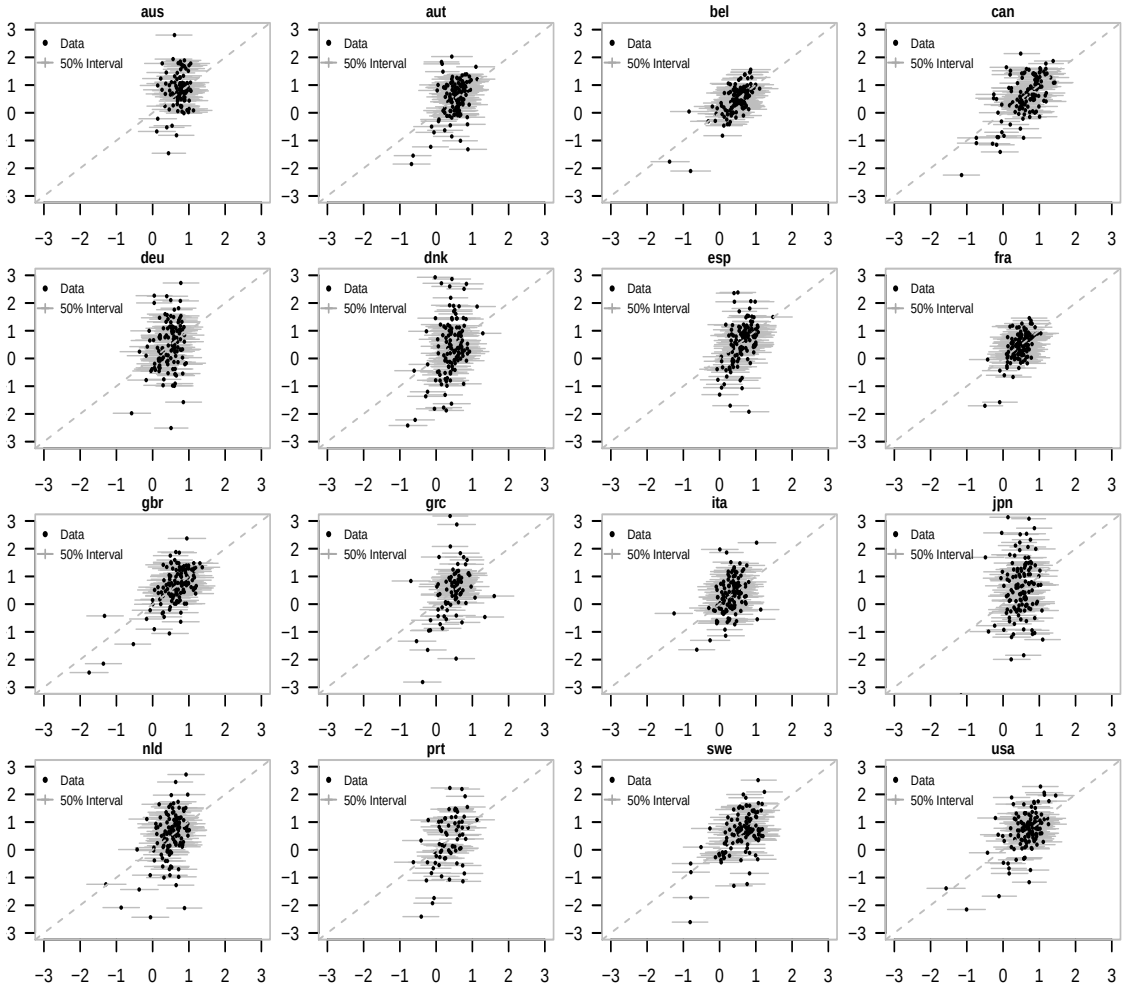

Figure 4: *The horizontal axis shows the predicted growth rate rates and the vertical axis shows the actual growth rates. The gray dotted line is the 45 degree line. The black dots are the points showing the predicted and actual growth rates and the gray lines are their respective 50% intervals.*

3.2 Model 2

$$\begin{aligned} p(y_{it}|\beta_{0i}, \beta_H, \beta_{yi}, \beta_{Ldi}, \beta_{Hdi}, \beta_{fs}, \beta_l) \sim \mathcal{N}(&\beta_{0i} + \beta_{Hi} + \beta_{yi} \cdot y_{it-1} \\ &+ \beta_{Ldi} \cdot d_{L_{it-1}} + \beta_{Hdi} \cdot d_{H_{it-1}} \\ &+ \beta_l \cdot lab_{it}, \sigma^2) \end{aligned} \tag{7}$$

$$p(\beta_0) \sim \mathcal{N}(\mu_0, \tau_0) \tag{8}$$

$$p(\beta_H) \sim \mathcal{N}(\mu_H, \tau_H) \tag{9}$$

$$p(\beta_y) \sim \mathcal{N}(\mu_y, \tau_y) \tag{10}$$

$$p(\beta_{Ld}) \sim \mathcal{N}(\mu_{Ld}, \tau_{Ld}) \tag{11}$$

$$p(\beta_{Hd}) \sim \mathcal{N}(\mu_{Hd}, \tau_{Hd}) \tag{12}$$

$$p(\beta_{fs}) \sim \mathcal{N}(\mu_{fs}, \tau_{fs}) \tag{13}$$

$$p(\beta_l) \sim \mathcal{N}(\mu_l, \tau_l) \tag{14}$$

Equation 7 is the likelihood function and equations 8 through 14 are priors on the parameters. This model is a replication of Proano et. al (2004). The debt to GDP ratio is taken conditional on 'Low' and 'High' financial stress. The thershold value for financial stress is 2.27. If financial stress is higher than that value for a country quarter, that period is regarded as one of high stress, else it is a period of low stress. The debt to GDP ratio for high and low stress periods are given respectively by $d_{H_{it-1}}$ and $d_{L_{it-1}}$. High stress periods have an intercept coefficient and a slope coefficient associated with them, denoted by β_{Hi} and β_{Hdi}.

Posterior Predictive Checks - Model 2

In Figure 5 we observe that the predictions of the model are inadequate and perform poorer than the simple linear regression model. Model 1 which did a reasonable job predicting growth rates in Belgium, Canada, France, Great Britain and U.S.A outperforms Model 2 here. So we continue modeling.

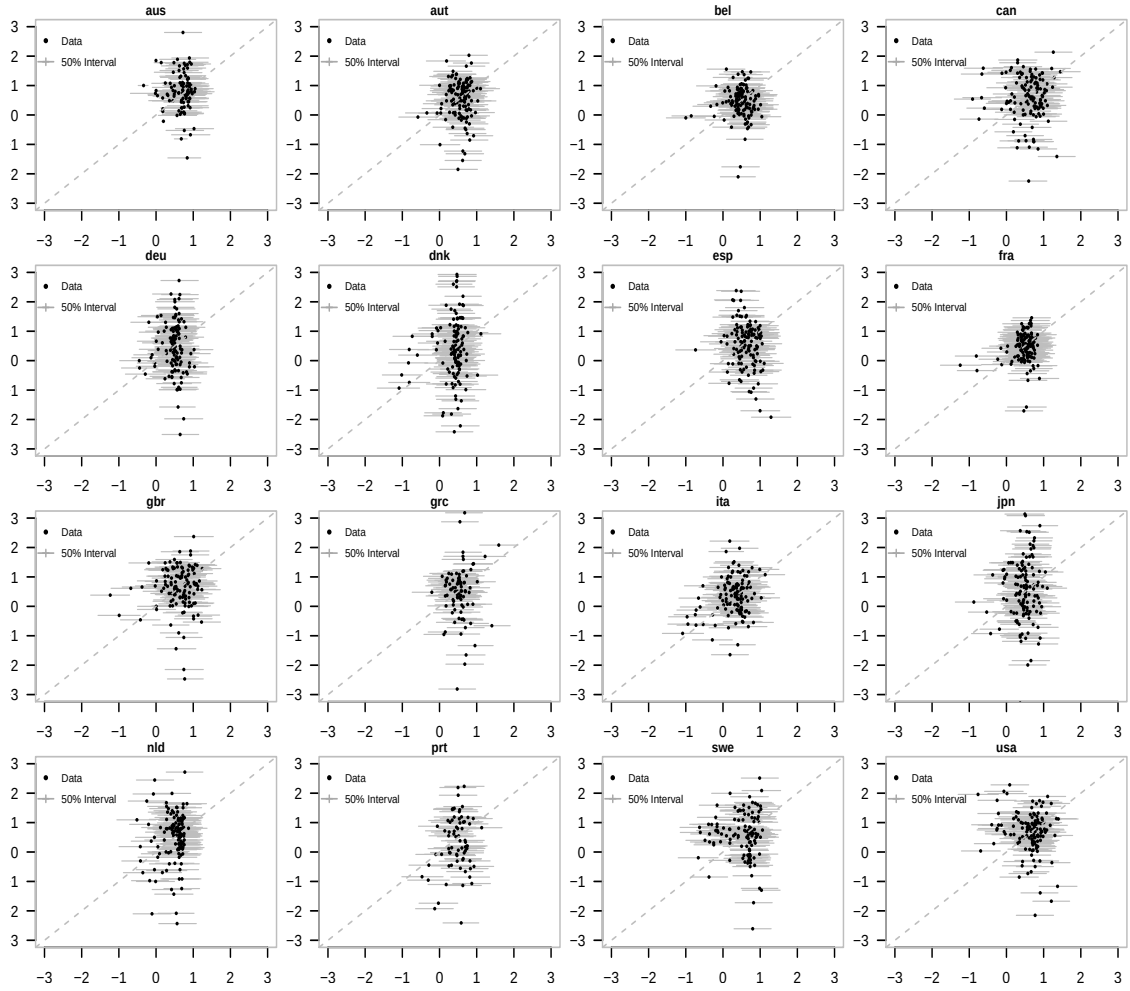

Figure 5: *The horizontal axis shows the predicted growth rate rates and the vertical axis shows the actual growth rates. The gray dotted line is the 45 degree line. The black dots are the points showing the predicted and actual growth rates and the gray lines are their respective 50% intervals.*

3.3 Model 3

Building on Proano et. al (2004), I consider lagged debt to GDP ratio for different values of financial stress and include financial stress explicitly as a predictor. The range of financial stress is approximately [-6,17]. Approximating Proano et. al (2004), I consider lagged debt to GDP ratio for financial stress levels greater than 0, 2, 6 and 10. These stress levels are chosen arbitrarily and the goal is to evaluate Proano et. al's claim that 2.27 is a thresholding value of financial stress. The purpose of this model is to examine whether the change that Proano et. al. claim occurs at 2.27, persists at higher levels of financial stress. They suggest that beyond financial stress of 2.27, debt to GDP ratio has a negative impact on growth rate of GDP.

$$
\begin{aligned}
p(y_{it}|\beta_{0i}, \beta_H, \beta_{yi}, \beta_{di}, \beta_{0di}, \beta_{2di}, \beta_{6di}, \beta_{10di}, \beta_{fs}, \beta_l) \sim \mathcal{N}(&\beta_{0i} + \beta_{yi} \cdot y_{it-1} + \beta_{di} \cdot d_{it-1} \\
&+ \beta_{0di} \cdot d_{0_{it-1}} + \beta_{2di} \cdot d_{2_{it-1}} + \beta_{6di} \cdot d_{6_{it-1}} \\
&+ \beta_{10di} \cdot d_{10_{it-1}} + \beta_{fs} \cdot fs_{it} + \beta_l \cdot lab_{it}, \sigma^2)
\end{aligned} \tag{15}
$$

$$p(\beta_0) \sim \mathcal{N}(\mu_0, \tau_0) \tag{16}$$

$$p(\beta_H) \sim \mathcal{N}(\mu_H, \tau_H) \tag{17}$$

$$p(\beta_y) \sim \mathcal{N}(\mu_y, \tau_y) \tag{18}$$

$$p(\beta_d) \sim \mathcal{N}(\mu_d, \tau_d) \tag{19}$$

$$p(\beta_{0d}) \sim \mathcal{N}(\mu_{0d}, \tau_{0d}) \tag{20}$$

$$p(\beta_{2d}) \sim \mathcal{N}(\mu_{2d}, \tau_{2d}) \tag{21}$$

$$p(\beta_{6d}) \sim \mathcal{N}(\mu_{6d}, \tau_{6d}) \tag{22}$$

$$p(\beta_{10d}) \sim \mathcal{N}(\mu_{10d}, \tau_{10d}) \tag{23}$$

$$p(\beta_{fs}) \sim \mathcal{N}(\mu_{fs}, \tau_{fs}) \tag{24}$$

$$p(\beta_l) \sim \mathcal{N}(\mu_l, \tau_l) \tag{25}$$

Equation 15 is the likelihood function and equations 16 through 25 are the priors on the parameters.

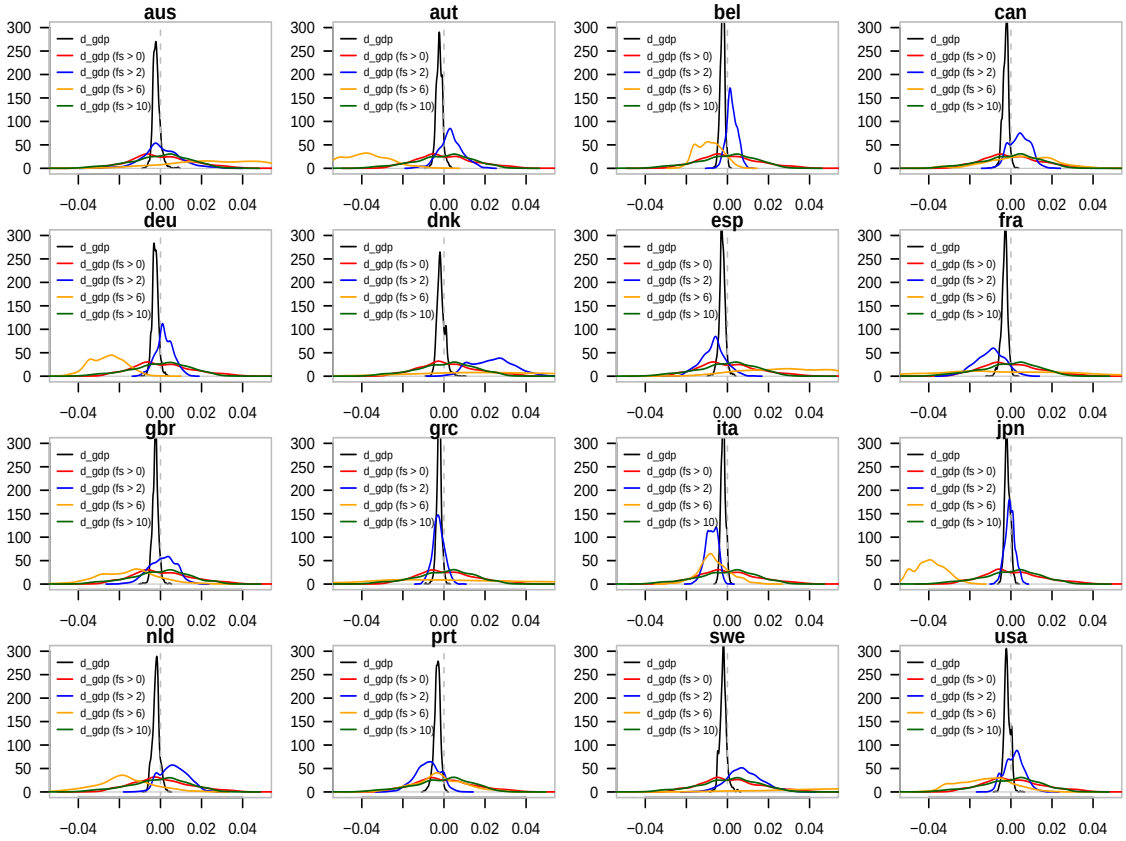

Figure 6: *The marginal posterior densities of* $\beta_d, \beta_{0d}, \beta_{2d}, \beta_{6d}, \beta_{10d}$ *in black, red, blue, orange and green respectively.*

It is interesting in Figure 6 that the posterior distribution of β_d (black) which is the coefficient associated with debt to GDP for the entire data is sharply centered at 0, with very little variance. The coefficient for β_{0d} (red) which corresponds to debt to GDP when financial stress greater that 0, is highly variant and appears to be roughly centered at 0, but due to the extremely high variance we cannot conclude anything. This trend is similar for the coefficients β_{6d} (orange) and β_{10d} (green). However the coefficient β_{2d} (blue) seems to be more volatile. In countries like Belgium and Canada the posterior mass is concentrated to the right of 0 and in countries like Italy, Spain and Portugal the posterior mass is to the left of 0. However there appears to be some trend in the data between financial stress levels 2 and 6. This is worth further investigation. Perhaps this is the trend that Proano et. al picked up on, but it seems to disappear at financial stress levels higher than 6. There could be some causal relationship here or it may simply be noise in the data.

Posterior Predictive Checks - Model 3

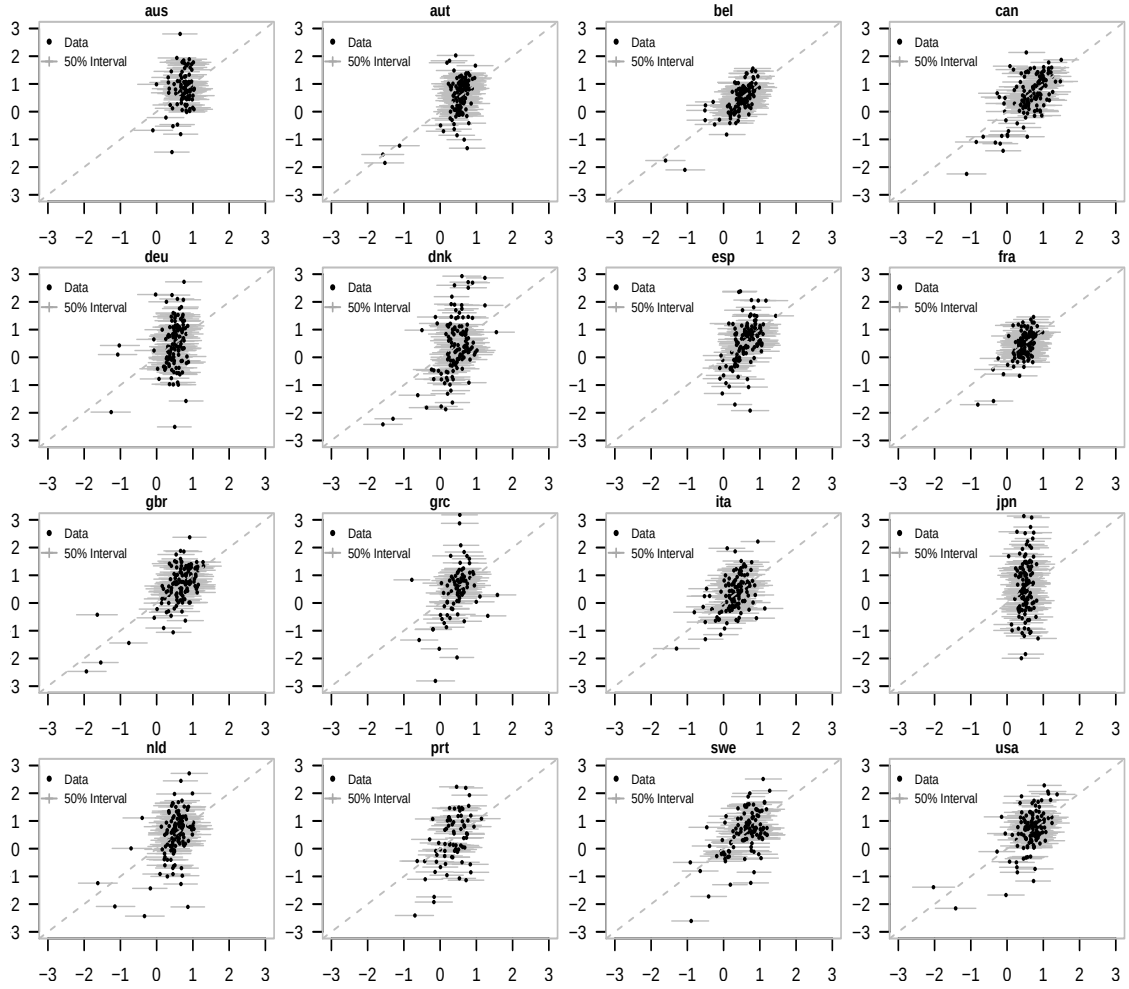

Figure 7: *The horizontal axis shows the predicted growth rate rates and the vertical axis shows the actual growth rates. The gray dotted line is the 45 degree line. The black dots are the points showing the predicted and actual growth rates and the gray lines are their respective 50% intervals.*

3.4 Model 4

Given that there is some trend in the debt to GDP data between financial stress levels 2 and 6, in Figure 8, I show another exploratory plot.

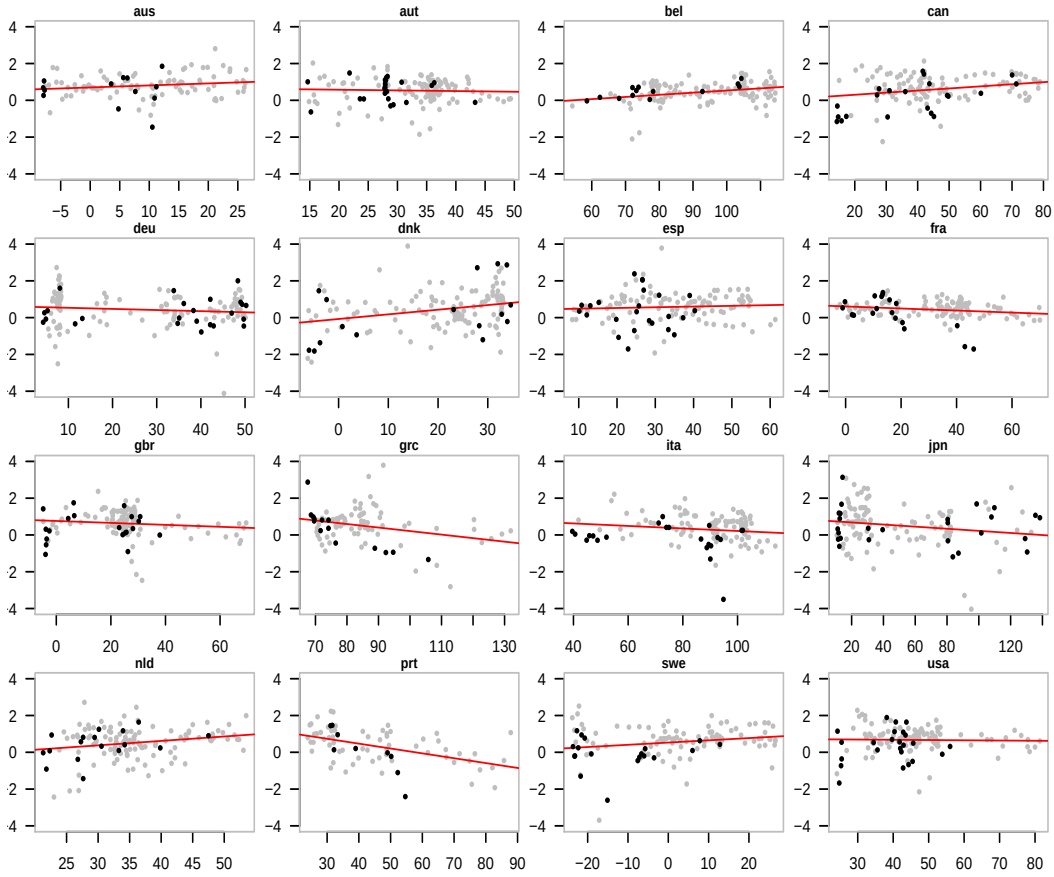

Figure 8: *The horizontal axis is the Debt to GDP Ratio. The vertical axis is the Growth Rate of GDP for each country under study. The red line is the OLS regression line. The black dots are the points corresponding to growth rate of GDP vs debt to GDP ratio for financial stress between 2 and 6. Gray dots are the remaining data at other stress levels.*

The aim of this plot is to visually represent the part of the data under closer investigation with the following model. The black dots show the relation between growth of GDP and debt to GDP ratio for financial stress levels between 2 and 6. We saw in the previous model that when financial stress is greater than 2, the marginal posterior distribution of β_{2d} moves away from 0, but this trend all but disappears at financial stress 6. So this model deals with this interval explicitly (i.e financial stress between 2 and 6).

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$$p(y_{it}|\beta_{0i},\beta_H,\beta_{yi},\beta_{di},\beta_{2di},\beta_{3di},\beta_{4di},\beta_{5di},\beta_{fs},\beta_l) \sim \mathcal{N}(\beta_{0i}+\beta_{yi}\cdot y_{it-1}+\beta_{di}\cdot d_{it-1} + \beta_{2di}\cdot d_{2_{it-1}}+\beta_{3di}\cdot d_{3_{it-1}}+\beta_{4di}\cdot d_{4_{it-1}} + \beta_{5di}\cdot d_{5_{it-1}}+\beta_{fs}\cdot fs_{it}+\beta_l\cdot lab_{it},\sigma^2) \tag{26}$$

$$p(\beta_0) \sim \mathcal{N}(\mu_0,\tau_0) \tag{27}$$

$$p(\beta_H) \sim \mathcal{N}(\mu_H,\tau_H) \tag{28}$$

$$p(\beta_y) \sim \mathcal{N}(\mu_y,\tau_y) \tag{29}$$

$$p(\beta_d) \sim \mathcal{N}(\mu_d,\tau_d) \tag{30}$$

$$p(\beta_{2d}) \sim \mathcal{N}(\mu_{2d},\tau_{2d}) \tag{31}$$

$$p(\beta_{3d}) \sim \mathcal{N}(\mu_{3d},\tau_{3d}) \tag{32}$$

$$p(\beta_{4d}) \sim \mathcal{N}(\mu_{4d},\tau_{4d}) \tag{33}$$

$$p(\beta_{5d}) \sim \mathcal{N}(\mu_{5d},\tau_{5d}) \tag{34}$$

$$p(\beta_{fs}) \sim \mathcal{N}(\mu_{fs},\tau_{fs}) \tag{35}$$

$$p(\beta_l) \sim \mathcal{N}(\mu_l,\tau_l) \tag{36}$$

Equation 26 is the likelihood function and equations 27 to 36 are the priors on the parameters.

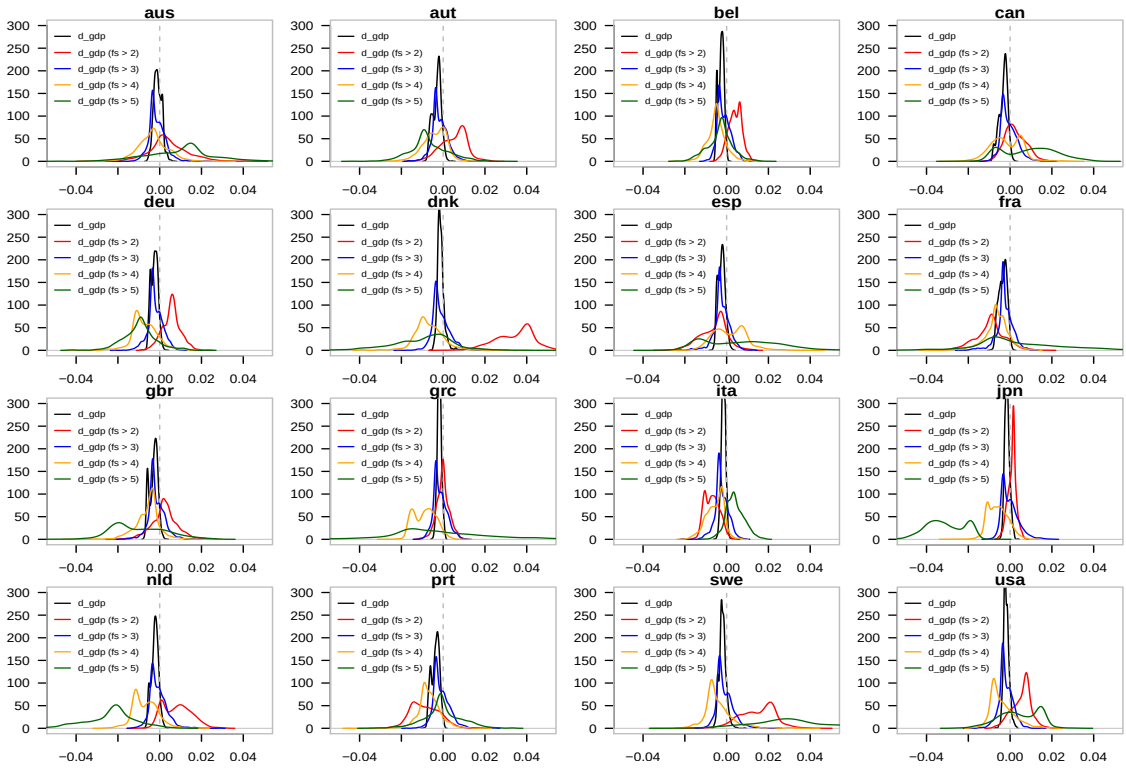

Figure 9: *The marginal posterior densities of* $\beta_d, \beta_{2d}, \beta_{3d}, \beta_{4d}, \beta_{5d}$ *in black, red, blue, orange and green respectively.*

This model examines the relationship between lagged debt to GDP ratio and growth level of GDP for financial stress at 2, 3, 4, 5. The red and blue lines represent the marginal posterior distributions of β_{2d} and β_{3d}. There seems to be some pattern in the data because as the financial stress levels increase to 4 and 5 the orange and green the posterior distributions become highly variable and there is no clear pattern discernible.

Posterior Predictive Checks - Model 4

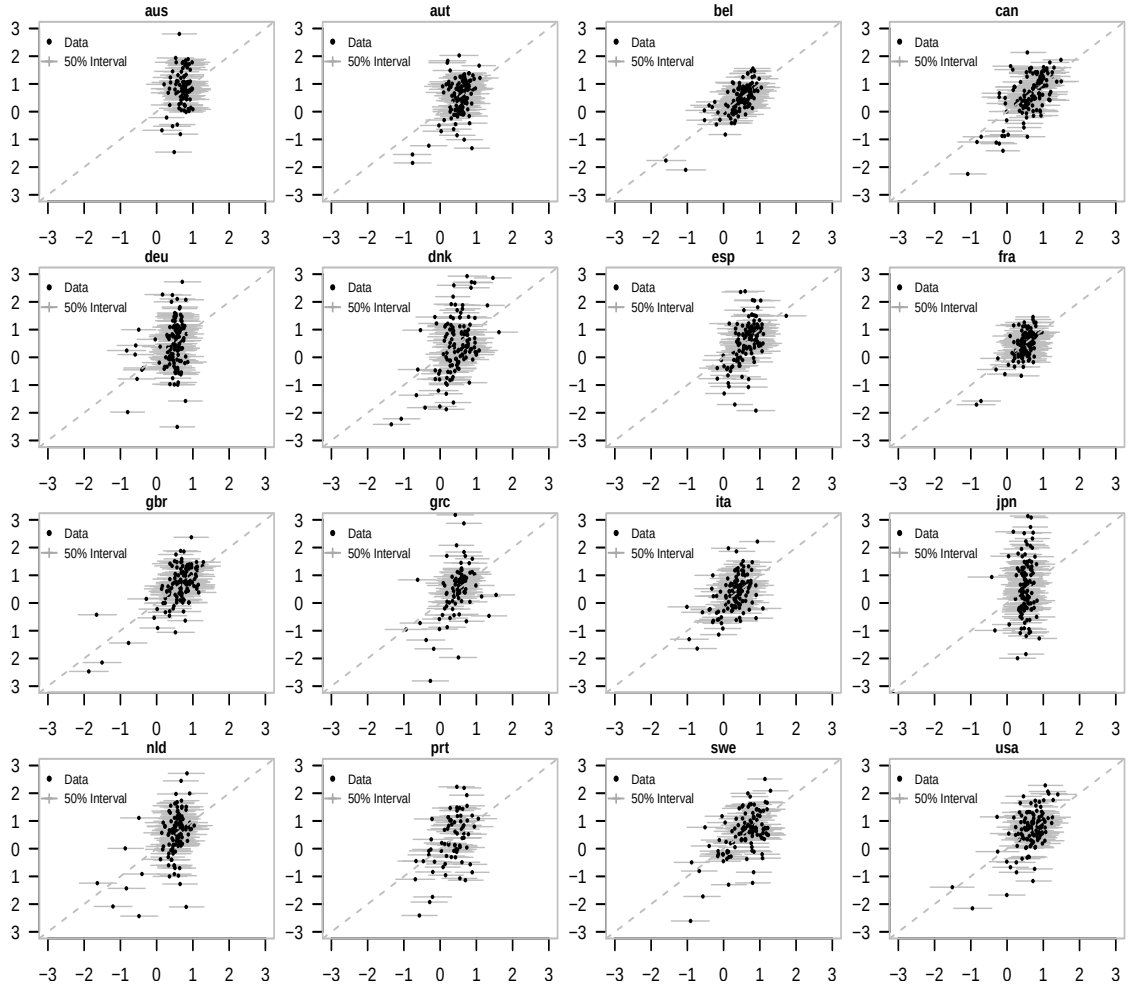

Figure 10: *The horizontal axis shows the predicted growth rate rates and the vertical axis shows the actual growth rates. The gray dotted line is the 45 degree line. The black dots are the points showing the predicted and actual growth rates and the gray lines are their respective 50% intervals.*

3.5 Model 5

This model builds on the previous models. I include new predictors as well as create a hierarchy for financial stress levels instead of including it as a separate predictor. I create 6 financial stress levels as depicted in the table below.

Table 1: Financial stress brackets

Financial Stress Level	Interval
1	$fs \in [-7, -3]$
2	$fs \in [-3, 0]$
3	$fs \in [0, 2]$
4	$fs \in [2, 4]$
5	$fs \in [4, 7]$
6	$fs \in [7, 18]$

The predictors in this model are growth lagged by one period, debt lagged by one period, labor force growth rate, exchange rate, rate of inflation and real interest rate.

$$\begin{aligned} p(y_{it}|\Theta) \sim \mathcal{N}(&\beta_0 + \beta_{fs} + \beta_{gr} \cdot y_{it-1} + \beta_{di} \cdot debt_{it-1} \\ &+ \beta_{lab} \cdot lab_{it} + \beta_{ex} \cdot exch_{it} + \beta_{inf} \cdot inf_{it} + \beta_{int} \cdot int_{it}, \sigma^2) \end{aligned} \tag{37}$$

where Θ is a vector of parameters including the hyperparameters. Equation 37 represents the likelihood function. The following equations show the priors on the parameters.

$$p(\beta_0) \sim \mathcal{N}(0, 0.01) \tag{38}$$
$$p(\beta_{fs}) \sim \mathcal{N}(\mu_{fs}, \tau_{fs}) \tag{39}$$
$$p(\beta_{gr}) \sim \mathcal{N}(\mu_{gr}, \tau_{gr}) \tag{40}$$
$$p(\beta_d) \sim \mathcal{N}(\mu_d, \tau_d) \tag{41}$$
$$p(\beta_l) \sim \mathcal{N}(0, 0.5) \tag{42}$$
$$p(\beta_{ex}) \sim \mathcal{N}(0, 0.5) \tag{43}$$
$$p(\beta_{inf}) \sim \mathcal{N}(0, 0.5) \tag{44}$$
$$p(\beta_{int}) \sim \mathcal{N}(0, 0.5) \tag{45}$$
$$p(\mu_{fs}) \sim \mathcal{N}(0, 0.001) \tag{46}$$
$$p(\tau_{fs}) \sim Cauchy(0, 0.0001) \tag{47}$$
$$p(\mu_d) \sim \mathcal{N}(0, 0.01) \tag{48}$$
$$p(\tau_d) \sim Cauchy(0, 0.0005) \tag{49}$$
$$p(\sigma) \sim Cauchy(0, 0.5) \tag{50}$$

The vector $\Theta = \{\beta_0, \beta_{fs}, \beta_{gr}, \beta_d, \beta_l, \beta_{ex}, \beta_{inf}, \beta_{int}, \mu_{fs}, \tau_{fs}, \mu_d, \tau_d, \sigma^2\}$.

Figure 11 shows the marginal posterior distributions of the 6 financial stress intercept corresponding to each level of stress. We observe that as financial stress increases its effect on growth (β_{fs}) shifts the marginal posterior towards 0 and then below it. Thus it has a greater negative impact on growth rate as financial stress increases. This tendency has been captured by previous literature.

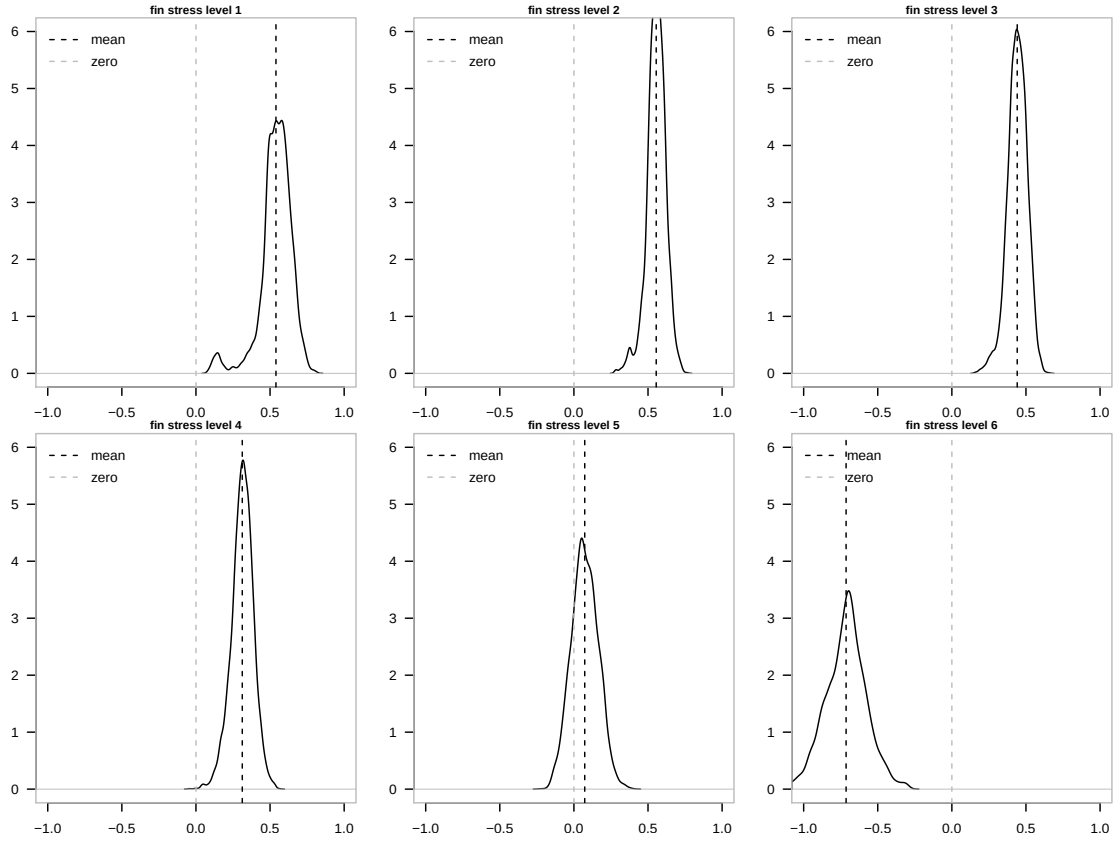

Figure 11: *The marginal posterior distribution of (β_{fs}). The distribution shifts leftward at higher levels of financial stress. The black dotted line shows the mean of the marginal posterior distribution of (β_{fs}) and the gray dotted line is at zero.*

Table 2 shows the values of the hyperparameters of the distribution of β_{fs}.

Table 2: Financial stress hyperparameters

Parameter	Value
μ_{fs}	0
τ_{fs}	0.569

Figure 12 shows the marginal posterior distribution of β_{di} where $i = 1, .., 16$ countries. Figure 13 is the posterior predictive check of growth rates.